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those also who are called upon in case of loss to adjust the claims, it is not saying at all too much to remark, that there are not a few who will be the better for a renewed and thoughtful attention to the points discussed. To this it may be added, that if all those whose duty it is to understand the subject, but who do not, will only give a patient hearing to what has now been advanced, there will be an audience quite numerous enough to satisfy the ambition of the writer.

On the Value of Policies of Assurance in connection with Life Interests. By T. B. SPRAGUE, M.A., F.I.A., Actuary to the Liverpool and London Insurance Company.

IN the course of an actuary's practice, the question not uncommonly arises—What is the value of a life interest, accompanied by a policy of assurance on the same life? The most obvious course for solving this question is to value the life interest and the policy separately—the former by the well-known formula, $\frac{1}{p+d}-1$.*

With regard to the policy, it is not so clear how it should be valued. If valued on the same principles as the life interest, the value of a policy for £1 would be $1-(p+d)(1+A)$;† but this formula is quite inapplicable, for it gives a negative value for many years to a policy, and almost always a much smaller value than the surrender value allowed by the Office. There is, therefore, apparently no choice but to take the value of the policy at the latter amount.

By taking the policy, however, in connection with the life interest, we are able to attach a more definite value to it, as I shall now proceed to show. The method I am going to describe is one that has doubtless occurred to many other persons who have had occasion to think over the question; but as the subject

* This formula was first given by Mr. Griffith Davies: it coincides with the less elegant one given in *Jones on Annuities*, art. 246, $s=a \cdot \frac{1-p(1+i)}{i+p(1+i)}$, which may be reduced to the form in the text by the help of the formula $d=\frac{i}{1+i}$. A proof of the formula, as well as of some very useful and practical extensions of it, is given by Mr. Jellicoe in the *Assurance Magazine*, vol. ii., p. 159.

† The value of a reversion, as given in Mr. Jellicoe's paper already referred to, is $1-d(1+A)$. In the case of a policy for £1, we must subtract from this value the cost of an annuity equal to the annual premium, or $p(1+A)$, giving the value of the policy as stated in the text, $1-(p+d)(1+A)$.

is one of practical interest, I think no apology is necessary for bringing it before the readers of the *Assurance Magazine*.

The purchaser of a life interest, in order to protect himself from loss by the premature death of the life on which it depends, must effect a policy of assurance on the life; and the same principles of valuation which fix the value of the life interest at $\frac{1}{p+d}-1$, show that the proper amount of the policy to be effected is $\frac{1}{p+d}$. Hence, if I be the value of a life interest of £1, the proper amount (S) of the policy to cover an annuity of £ a , is $(1+I)a$; and conversely, a policy for the sum S will cover an annuity of $\frac{S}{1+I}$, or $S(p+d)$. Supposing a policy to be effected for the full amount $\frac{1}{p+d}$, it is seen that after the lapse of any number of years, the value of the life interest and policy together will always be the same, viz., $\frac{1}{p+d}-1$; the annuity providing for the premium of insurance and the interest on the outlay during the continuance of the life, and the policy returning the capital on the extinction of the life. It is thus seen that the policy here has a definite value, which is in fact the difference between the original value of the life interest and its value at the advanced age. I shall give further on a general expression for this value.

The value of a life interest and of the *corresponding policy* being thus easily found, we are led to the following rule for valuing a life interest only partially covered by a policy:—"Determine how much of the annuity is covered by the policy, and find the value of that portion and of the policy together, as shown above; then value the remainder of the annuity in the ordinary way."

To take an example:—Suppose it is required to find the value of an annuity of £100 during the continuance of a life that can now be insured at £3. 10s. per cent, together with a policy for £500 on the same life, effected some years back at £2. 10s. per cent. The value of a life interest of £1, when the premium of insurance is £2. 10s., will, by the formula already quoted, be £12.770 (d being taken equal to .04762, which supposes that the purchaser makes 5 per cent on his outlay). The policy to cover the interest completely should be for the sum of £13.770; hence, by proportion, the policy for £500 will cover an annuity of £36.31, and the value of this annuity and the policy together will be £500—£36.31, or

£463·69. At the increased premium £3. 10s., the value of a life interest of £1 will be £11·1036, and the value of the remaining annuity of £63·69 will therefore be £707·188. Adding to this amount the value found above, we find the value of the proposed life interest and policy together to be £1170·878.

If the life interest were valued without the policy, its value would be only £1110·360; so that the value of the policy, considered in conjunction with the life interest, is £60·518, which is equal to the difference between the value of an annuity of £36·31 when p is taken equal to ·035, and the value of the same annuity when p is ·025.

What precedes will, I think, be sufficient to show how the case proposed is to be dealt with: I will now proceed to obtain algebraical expressions for the values of the policy and of the annuity. Suppose that a policy is effected at the rate p_1 to cover a life interest of £1, and that after the lapse of n years the premium is p_2 ; then the original value of the life interest being $\frac{1}{p_1+d} - 1$, this will be the value of the life interest and policy together at the end of n years; but the value of the former after n years will be $\frac{1}{p_2+d} - 1$; therefore the value of the policy after n years is $\frac{p_2-p_1}{(p_1+d)(p_2+d)}$. Again, since the amount of the policy is $\frac{1}{p_1+d}$, the value of a policy for £1 will be $\frac{p_2-p_1}{p_2+d}$; or, if I denote the value of the life interest of £1 at the increased age, since $I = \frac{1}{p_2+d} - 1$, the value of the policy for £1 will be

$$(p_2-p_1)(1+I) \quad . \quad . \quad . \quad (1)$$

It will be noticed that this expression is in close analogy to the well-known one for the value of a policy—

$$(p_{m+n}-p_m)(1+a_{m+n}) \quad . \quad . \quad . \quad (2)$$

given in *Jones on Annuities*, art. 254; but it must be remembered that p_m and p_{m+n} here denote the *net* premiums, while in our formula p_1 and p_2 denote those actually payable.

It may also be remarked, that the value of a policy to the holder, who wishes to continue it, is

$$(p_2-p_1)(1+A) \quad . \quad . \quad . \quad (3)$$

where A is now the value of an annuity at the price at which he

could purchase it from a respectable Company, and may be taken at Carlisle $3\frac{1}{2}$ per cent.

The formula (1) gives the value of a policy for £1 when considered in connection with the corresponding life interest. For the general case proposed we proceed as follows:—Suppose the life interest is £A per annum, and that a policy for £S has been effected on the life at the premium p_1 for £1; also that the premium that would have to be paid now is p_2 for £1. At the original premium the value of a life interest of £1 is $\frac{1}{p_1+d}-1$, and the policy to cover it would be for the amount $\frac{1}{p_1+d}$; hence a policy for S would cover a life interest of $S(p_1+d)$, and the value of the life interest and policy together will be $S(1-p_1-d)$. The portion of the annuity not covered by the policy is $A-S(p_1+d)$, and the value of it is $\left\{A-S(p_1+d)\right\} \left\{\frac{1}{p_2+d}-1\right\}$. Adding to this the former value, $S(1-p_1-d)$, the value of the life interest and policy together is

$$\begin{aligned} & A\left(\frac{1}{p_2+d}-1\right) - \frac{p_1+d}{p_2+d}S + S \\ &= A\left(\frac{1}{p_2+d}-1\right) + \frac{p_2-p_1}{p_2+d}S. \end{aligned}$$

Let I denote the value of the life interest when the premium is p_2 , then $I = \frac{1}{p_2+d}-1$, and the expression last written down becomes

$$AI + S(p_2-p_1) \cdot (1+I). \quad (4)$$

In this result, AI is the value of the life interest at the advanced age. The other term of the value is the amount by which the value of the life interest is increased, in consequence of having the policy of assurance attached to it; and is at once seen to be the value of a policy for the sum S, calculated by the formula (1). Hence, in place of the rule given previously, we may adopt the following, which indeed might have been anticipated:—"Value the life interest in the ordinary way, and add to the result the value of the policy calculated by the formula (1)."

This will be a rather simpler course in practice. Thus, to take our former example,

$$p_1 = \cdot 025, \quad p_2 = \cdot 035, \quad I = \frac{1}{p_2+d}-1 = 11\cdot 1036;$$

so that the value of the policy is $S(p_2-p_1)(1+I) = 60\cdot 518$, which, added to AI or £1110·360, gives the value of the annuity

and policy together, £1170·878, as before. It will be interesting to compare the value of the policy as thus found with the ordinary value. Suppose that it was taken out at the age 36, and has been 10 years in force, then the value by the Carlisle Table, at 3 per cent, is £67·715. The value to the holder, as given by the formula (3), is £78·485.

In all that precedes, I have supposed that the policy is smaller than is required to cover the life interest entirely. If the contrary is the case, and the policy is in excess of the amount required, we shall have to determine how much of it is required to secure the life interest; and the value of this portion and of the life interest is determined in the manner pointed out above. The remaining part of the policy must be valued separately, and it will in general be most practical to estimate it at the Office value; since on the one hand, it must not be taken at a less value than can actually be obtained for it, and on the other hand, it will not in general be worth more to a purchaser.

I have thought it would be a useful addition to this paper to give a table to facilitate the valuation of life interests and contingent reversionary annuities, by means of the formula $\frac{1}{p+d} - 1$.

Such a table has already been given by Mr. Griffith Davies, *viz.*, No. XXV in his work on Life Assurance. He has there tabulated the value of a life interest for all ages from 21 to 65, for the three cases in which the purchaser makes 5, 6, and 7 per cent on his outlay, and on the supposition that the premiums for the assurance are those charged by the "Equitable" Society. But in practice these will not generally be the rates charged; and indeed, the premiums charged by the various Companies differ so widely, that it would be impossible to select any table of premiums that would meet with general approval. In valuing a life interest, each actuary will probably either use the non-participating rates charged by his own Company, or some average rate deduced from the premiums of the Companies with which he would effect the necessary re-assurances. For this reason, it has seemed the more desirable course to tabulate the value of the quantity $\frac{1}{p+d} - 1$ for all values of the premium for £100, from £1. 10s. to £8., at intervals of 1s., these being the extreme premiums that are likely to occur in practice. Then, the proper premium for any age being fixed upon, the value of the life interest is given by the table, when the premium is an exact number of pounds and shillings.

This value is calculated on the supposition that the purchaser makes 5 per cent on his outlay, as I believe that to be the rate of interest which is almost invariably used in these transactions. For intermediate rates, an interpolation is necessary, which may be performed mentally; and to assist this, the differences between successive values are given in the column headed Δ . Thus, supposing the rate to be £2. 12s. 4d., the value of an annuity when the premium is £2. 12s. is £12·583, and the difference in this value for a difference of 1s. in the premium being ·091, the difference for 4d. will be ·030, which must be subtracted, making the value of the annuity £12·553. When the number of pence in the premium is greater than six, it will be more convenient to take the higher premium and make an additive correction. Thus, supposing the premium to be £4. 5s. 10d., the annuity for £4. 6s. is £10·035, to which must be added ·010 for the 2d. by which the assigned premium falls short of £4. 6s., making the value required £10·045.

The values have been calculated by the help of Barlow's Table of Reciprocals, taking in the first instance $d = \cdot04762$, and making the requisite correction when the difference between the above and the true value $\cdot04761905$ would affect the third decimal place of the annuity, so that the annuities are in all cases exact to the third decimal place. The irregularities in the differences, as seen in the column headed Δ , arise from the last figure in the value of the annuity being increased by unity when the next succeeding figure was 5 or greater than 5. These irregularities would of course disappear if the values were given to four decimal places; but besides the increased labour of forming the table in that case, the process of interpolation for intermediate rates would then be much more troublesome, so that it is thought the table in the form here given will be quite sufficient for practical purposes.

It may be noticed, that the value of the life interest, in terms of the premium of insurance, may be found by using *inversely* Orchard's Tables, which give the annual premium in terms of the annuity. Thus, in the table corresponding to 5 per cent interest, to the annuity 8·040 corresponds the premium £6·300, consistently with the result as seen in the subjoined table. That this ought to be the case is apparent, when we remember that Mr. Orchard has tabulated the function $\frac{1}{1+a_m} - d$. Putting this equal to p , and

solving for a_m , we have $a_m = \frac{1}{p+d} - 1$, which is the quantity

tabulated below. It is true, that in Mr. Orchard's tables p and a_m have meanings attached to them different to those we now assign, but the relation between the quantities is just the same. It will not, however, be found convenient in practice to use Mr. Orchard's tables for this purpose. These remarks sufficiently explain the object of the subjoined table.

Table showing the Value of a Life Interest, so as to allow the Purchaser 5 per cent on the outlay, for different values of the Premium of Insurance.

Premium to insure £100.	Value of Life Interest.	Δ.	Premium to insure £100.	Value of Life Interest.	Δ.
£ s. d.			£ s. d.		
1 10 0	14·970	127	3 6 0	11·404	76
1 11 0	14·843	124	3 7 0	11·328	76
1 12 0	14·719	123	3 8 0	11·252	75
1 13 0	14·596	121	3 9 0	11·177	73
1 14 0	14·475	118	3 10 0	11·104	73
1 15 0	14·357	118			
			3 11 0	11·031	72
1 16 0	14·239	115	3 12 0	10·959	71
1 17 0	14·124	113	3 13 0	10·888	70
1 18 0	14·011	112	3 14 0	10·818	70
1 19 0	13·899	110	3 15 0	10·748	68
2 0 0	13·789	109			
			3 16 0	10·680	68
2 1 0	13·680	107	3 17 0	10·612	67
2 2 0	13·573	105	3 18 0	10·545	66
2 3 0	13·468	104	3 19 0	10·479	66
2 4 0	13·364	103	4 0 0	10·413	65
2 5 0	13·261	101			
			4 1 0	10·348	64
2 6 0	13·160	99	4 2 0	10·284	63
2 7 0	13·061	98	4 3 0	10·221	63
2 8 0	12·963	97	4 4 0	10·158	62
2 9 0	12·866	96	4 5 0	10·096	61
2 10 0	12·770	94			
			4 6 0	10·035	60
2 11 0	12·676	93	4 7 0	9·975	60
2 12 0	12·583	91	4 8 0	9·915	59
2 13 0	12·492	91	4 9 0	9·856	59
2 14 0	12·401	89	4 10 0	9·797	58
2 15 0	12·312	88			
			4 11 0	9·739	57
2 16 0	12·224	87	4 12 0	9·682	57
2 17 0	12·137	85	4 13 0	9·625	56
2 18 0	12·052	85	4 14 0	9·569	56
2 19 0	11·967	84	4 15 0	9·513	55
3 0 0	11·883	82			
			4 16 0	9·458	54
3 1 0	11·801	82	4 17 0	9·404	54
3 2 0	11·719	80	4 18 0	9·350	53
3 3 0	11·639	79	4 19 0	9·297	53
3 4 0	11·560	79	5 0 0	9·244	52
3 5 0	11·481	77			

Table showing the Value of a Life Interest—continued.

Premium to insure £100.	Value of Life Interest.	Δ.	Premium to insure £100.	Value of Life Interest.	Δ.
£ s. d.			£ s. d.		
5 1 0	9·192	52	6 11 0	7·840	39
5 2 0	9·140	51	6 12 0	7·801	38
5 3 0	9·089	51	6 13 0	7·763	38
5 4 0	9·038	50	6 14 0	7·725	38
5 5 0	8·988	50	6 15 0	7·687	38
5 6 0	8·938	49	6 16 0	7·649	37
5 7 0	8·889	48	6 17 0	7·612	37
5 8 0	8·841	49	6 18 0	7·575	37
5 9 0	8·792	47	6 19 0	7·538	36
5 10 0	8·745	47	7 0 0	7·502	36
5 11 0	8·698	47	7 1 0	7·466	36
5 12 0	8·651	47	7 2 0	7·430	35
5 13 0	8·604	46	7 3 0	7·395	35
5 14 0	8·558	45	7 4 0	7·360	35
5 15 0	8·513	45	7 5 0	7·325	34
5 16 0	8·468	45	7 6 0	7·291	35
5 17 0	8·423	44	7 7 0	7·256	34
5 18 0	8·379	44	7 8 0	7·222	33
5 19 0	8·335	43	7 9 0	7·189	34
6 0 0	8·292	43	7 10 0	7·155	33
6 1 0	8·249	43	7 11 0	7·122	33
6 2 0	8·206	42	7 12 0	7·089	32
6 3 0	8·164	41	7 13 0	7·057	33
6 4 0	8·123	42	7 14 0	7·024	32
6 5 0	8·081	41	7 15 0	6·992	31
6 6 0	8·040	41	7 16 0	6·961	32
6 7 0	7·999	40	7 17 0	6·929	31
6 8 0	7·959	40	7 18 0	6·898	31
6 9 0	7·919	40	7 19 0	6·867	31
6 10 0	7·879	39	8 0 0	6·836	—

*On the Calculus of Finite Differences, and its Application to Problems in the Doctrine of Compound Interest and Certain Annuities. By WILLIAM CURTIS OTTER, F.R.A.S.**

(Continued from p. 340, vol. vii.)

THE idea of periodic or discontinuous functions was primitively introduced by Euler, and has since been the subject of extended investigation by M. Fourier, who has made some new and important applications of it in his mathematical theory of heat.

* Our attention has been called, by Mr. Benjamin F. Stevens, Secretary of the New England Mutual Life Insurance Company, at Boston, U.S., to the fact that the substance